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A Codicil To Massless Gauge Superfields of Higher Integer Superspins ¹

S. James Gates, Jr.² and K. Koutrolikos³

*Center for String and Particle Theory
Department of Physics, University of Maryland
College Park, MD 20742-4111 USA*

ABSTRACT

We study theories of 4D, $\mathcal{N} = 1$ supersymmetric massless, arbitrary integer superspins. A new state-of-the art is being established by the discovery of a new series of such theories for arbitrary superspin Y ($Y = s$ for any integer s) The lowest member of the series is surprisingly found to be a previously established formulation of the spin $(3/2, 1)$ supermultiplet.

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¹Supported in part by National Science Foundation Grant PHY-.

²gatess@wam.umd.edu

³koutrol@umd.edu

1 Introduction

The current state-of-the-art understanding on the subject of 4D, $\mathcal{N} = 1$ integer higher spin supersymmetric multiplets was established in a work by Kuzenko and Sibiryakov [1] (KS) wherein they gave two such formulations for each and every possible value of the integer superspin Y . These formulations are based on the introduction of constrained compensating superfields. This seminal work laid a foundation for a number of latter studies [2]. The goal of this work is to re-examine these schemes in order to be able to reproduce their results and, if possible, to discover new formulations in the case of integer superspins. This is exactly what will happen in the following. Their results will emerge naturally from our algorithm as a possible way a theory of higher, integer massless superspins can be formulated. Also we will discover the KS description is not the only consistent formulation and at least one alternative exists.

We approach the problem from a different angle, by studying actions for spinorial superfields. This is supported from the naive observation that for a massless, integer higher superspin theory the higher spin projection operator acting on the main superfield of the theory must give rise to a chiral object with an even number of indices.

$$(\Pi\Psi)_{\alpha(2s-1)} \propto D^{\alpha_{2s}} \bar{D}^2 D_{(\alpha_{2s}} \partial^{\dot{\alpha}_1}_{\alpha_{2s-1}} \dots \partial^{\dot{\alpha}_{s-1}}_{\alpha_{s+1}} \Psi_{\alpha(s)\dot{\alpha}(s-1)} \quad . \quad (1)$$

Therefore the main superfield of the theory must have an odd number of indices. Hence the construction of a higher integer superspin theory must be developed around a spinorial superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$.

2 The Setup

As our starting point, we assume the main object of the theory to be a spinorial superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$, this means it's highest spin component (the $\theta\bar{\theta}$ term) must be a propagating fermion. Therefore the mass dimensions of Ψ must be $1/2$. The most general action which is quadratic for a spinorial superfield $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ with mass

dimensions $[\Psi] = 1/2$ has to include exactly 2 D's (\bar{D} 's) and thus takes the form:

$$\begin{aligned}
S = \int d^8 z \Big\{ & c_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + c_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \Big\} .
\end{aligned} \tag{2}$$

Since this action is going to describe a massless supermultiplet, it should be invariant under a gauge symmetry. The most general gauge symmetry allowed by the invariance of the higher spin projection operator (1) has the following structure:

$$\delta \Psi_{\alpha(s)\dot{\alpha}(s-1)} = \left[\frac{1}{s!} \right] D_{(\alpha_s} K_{\alpha(s-1))\dot{\alpha}(s-1)} + \left[\frac{1}{(s-1)!} \right] D_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2))} . \tag{3}$$

The change of the above action with respect to the gauge transformation is:

$$\begin{aligned}
\delta S = \int d^8 z \Big\{ & (-2c_1 D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + a_2 \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)}) D^\beta \bar{D}_{\dot{\alpha}_{s-1}} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \\
& + 2c_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 D_{\alpha_s} K_{\alpha(s-1)\dot{\alpha}(s-1)} - a_1 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} D^2 \bar{D}_{\dot{\alpha}_s} K_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \left(a_2 \left[\frac{s+1}{s} \right] - a_1 \right) \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D^2 K_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + a_1 \left[\frac{s-1}{s} \right] \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} D_{\alpha_{s-1}} \bar{D}_{\dot{\alpha}_s} D^\beta K_{\beta\alpha(s-2)\dot{\alpha}(s-1)} + c.c. \Big\} .
\end{aligned} \tag{4}$$

This expression will play a key role in discovering the different ways an integer higher spin superspin theory can be realized. Our ultimate goal is, based on the above equations (3) and (4), to find all possible ways to build a gauge invariant action which on-shell has exactly the degrees of freedom to form a massless irreducible representation of the super Poincare group.

Following this path suggests the special case of $s = 1$ has to be treated separately since the index structure drastically changes⁴. At this point we focus on the $s > 1$ case

⁴The structure of the Λ term in the gauge transformation (3) has to change and equation (4) is simplified considerably.

3 Consideration for $s > 1$ Case

Now all we have to do is to find and introduce a set of appropriate and unconstrained compensators that will serve a double purpose. First of all they must give rise to a gauge invariant action and secondly this action on-shell must generate an irreducible representation of the super Poincare group. The second requirement, which is very restrictive, can be phrased in a different way. We can interpret it to say the invariant action constructed out of the superfield Ψ and a set of compensators when expanded in component fields must give the massless integer spin Fronsdal action for its bosonic piece and the massless half integer Fronsdal action for its fermionic piece. Since $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$ doesn't include all the fields needed for the Fronsdal actions, we need at least one propagating compensator. In principle it can be either a bosonic superfield⁵ or a fermionic superfield⁶.

Any attempt to introduce a fermionic compensator which has less indices than the main superfield appears precluded by the gauge invariance of the action. In order to succeed with the fermionic compensator it must be constrained⁷. Therefore the propagating compensator needed has to be a bosonic one V with zero mass dimensions. This follows the pattern of the higher spin theories developed so far in that the statistics flip between the main superfield and the propagating prepotential

One more very important observation is that V , must have a gauge transformation that involves either K or Λ parameters, since these are the only two available. But these parameters also have mass dimensions 0. That means that the gauge transformation will be algebraic (no derivatives are present). Ignoring the index structure and coefficients this gauge variation must look as

$$\delta V \sim K + \Lambda \quad .$$

This is not acceptable because it means that V can be *completely* gauged away and therefore there is nothing around to provide the extra degrees of freedom needed in order to form on-shell an irreducible multiplet. The only way out is if we allow the gauge parameters K or Λ to have some more D-structure within them. In this way we could introduce a zero mass dimensions bosonic compensator with a gauge transformation which is not algebraic and therefore could be used to gauge away all the unwanted degrees of freedom.

⁵This must possess a mass dimensions 0.

⁶This possesses mass dimensions 1/2.

⁷The constraint can be solved in terms of an unconstrained bosonic superfield.

The last piece of information that we need in order to construct the full theory, is what type of gauge transformations for V , should we introduce? The structure of the Fronsdal action comes to the rescue. In the massless integer spin Fronsdal action, there are two real bosonic component fields:

$$\text{the main field } h_{\alpha(s)\dot{\alpha}(s)}, \quad [h_{\alpha(s)\dot{\alpha}(s)}] = 1, \quad \delta h_{\alpha(s)\dot{\alpha}(s)} = \left[\frac{1}{s!2} \right] \partial_{(\alpha_s(\dot{\alpha}_s \zeta_{\alpha(s-1)\dot{\alpha}(s-1))} \quad ,$$

$$\text{a compensator } h_{\alpha(s-2)\dot{\alpha}(s-2)}, \quad [h_{\alpha(s-2)\dot{\alpha}(s-2)}] = 1, \quad \delta h_{\alpha(s-2)\dot{\alpha}(s-2)} = \partial^{\dot{\alpha}_s \alpha_s} \zeta_{\alpha(s-1)\dot{\alpha}(s-1)} \quad .$$

The main superfield Ψ can provide a component field with the index structure of $h_{\alpha(s)\dot{\alpha}(s)}$, but not one for the role of $h_{\alpha(s-2)\dot{\alpha}(s-2)}$. This field has to come from the compensator V . So on-shell V must provide only one real bosonic component with the proper index structure, mass dimensions and gauge transformation in order to play the role of $h_{\alpha(s-2)\dot{\alpha}(s-2)}$. This suggest that:

- V should be real and therefore it's index structure must be $V_{\alpha(s-1)\dot{\alpha}(s-1)}$,
- The $V_{\alpha(s-1)\dot{\alpha}(s-1)}^{(0,0)}$ component must be able to be gauged away, it has wrong mass dimensions and index structure. This can be achieved if:

$$\delta V_{\alpha(s-1)\dot{\alpha}(s-1)}| \sim \text{some component of the gauge parameter (algebraically),}$$

- The $V_{\alpha(s-1)\dot{\alpha}(s-1)}^{(2,0)}$ must be able to be gauged away (wrong index structure)

$$D^2 \delta V_{\alpha(s-1)\dot{\alpha}(s-1)}| \sim \text{some component of the gauge parameter (algebraically),}$$

- The $V_{\alpha(s)\dot{\alpha}(s)}^{(1,1)(S,S)}$ component must be able to be gauged away (wrong index structure)

$$[D_{(\alpha_s}, \bar{D}_{(\dot{\alpha}_s}] \delta V_{\alpha(s-1)\dot{\alpha}(s-1)}| \sim \text{some component of the gauge parameter (algebraically), and}$$

- The $V_{\alpha(s-2)\dot{\alpha}(s-2)}^{(1,1)(A,A)}$ component must survive on-shell and transform like

$$[D^{\alpha_{s-1}}, \bar{D}^{(\dot{\alpha}_{s-1}}] \delta V_{\alpha(s-1)\dot{\alpha}(s-1)}| \sim \partial^{\dot{\alpha}_{s-1} \alpha_{s-1}} \text{some component of the gauge parameter.}$$

These requirements fix the desired gauge transformation for the bosonic compensator to the following form:

$$\delta V_{\alpha(s-1)\dot{\alpha}(s-1)} = D^{\alpha_s} U_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{U}_{\alpha(s-1)\dot{\alpha}(s)} \quad . \quad (5)$$

Now it is very clear for what we are searching. By studying equation (4), we will explore all possible ways that we can introduce a real bosonic compensator with the above gauge transformation (5). This can happen only by a couple of ways. The first thing that is clear is that the parameter Λ , because of its index structure and the way it appears in (4), can not have an internal structure such it that will lead to the introduction of the desired compensator. So our efforts must focus on the parameter K . That also means that if we insist on having a Λ -term in (3) we must introduce another compensator which must be auxiliary⁸ or the coefficients related to the Λ terms must vanish.

3.1 The KS-series

By observing (4) we see that if $K_{\alpha(s-1)\dot{\alpha}s-1} = D^{\alpha s} U_{\alpha(s)\dot{\alpha}(s-1)}$ then the last two terms vanish and the change of the action becomes

$$\begin{aligned} \delta S = \int d^8 z & \left(-2c_1 D_{\alpha s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + a_2 \bar{D}_{\dot{\alpha}s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) D^\beta \bar{D}_{\dot{\alpha}s-1} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \\ & + \left(2c_2 D_{\alpha s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} - a_1 \bar{D}_{\dot{\alpha}s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) D^\beta U_{\beta\alpha(s-1)\dot{\alpha}(s-1)} \\ & + c.c. \end{aligned} \quad (6)$$

Now it is obvious that if

$$-2c_1 = a_2 \quad , \quad 2c_2 = -a_1 \quad , \quad (7)$$

the variation in (6) becomes

$$\begin{aligned} \delta S = \int d^8 z & \left[a_2 \left(D_{\alpha s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) \left[D^\beta \bar{D}_{\dot{\alpha}s-1} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \right] \right. \\ & \left. - a_1 \left(D_{\alpha s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) \left[D^\beta U_{\beta\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \right] \right] . \end{aligned} \quad (8)$$

The observations above suggest the introduction of two real compensators $B_{\alpha(s-1)\dot{\alpha}(s-1)}$, $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ with mass dimensions $[B] = 1$, $[V] = 0$ and gauge transformations

$$\delta B_{\alpha(s-1)\dot{\alpha}(s-1)} = \left[\frac{1}{(s-1)!} \right] \left(D^{\alpha s} \bar{D}_{(\dot{\alpha}s-1)} \Lambda_{\alpha(s)\dot{\alpha}(s-2)} + \bar{D}^{\dot{\alpha}s} D_{(\alpha s-1)} \bar{\Lambda}_{\alpha(s-2)\dot{\alpha}(s)} \right) , \quad (9)$$

$$\delta V_{\alpha(s-1)\dot{\alpha}(s-1)} = D^{\alpha s} U_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}s} \bar{U}_{\alpha(s-1)\dot{\alpha}(s)} .$$

The gauge transformation of Ψ becomes

$$\delta \Psi_{\alpha(s)\dot{\alpha}(s-1)} = -D^2 U_{\alpha(s)\dot{\alpha}(s-1)} + \left[\frac{1}{(s-1)!} \right] \bar{D}_{(\dot{\alpha}s-1)} \Lambda_{\alpha(s)\dot{\alpha}(s-2)} . \quad (10)$$

⁸in order not to introduce new degrees of freedom

In order to construct a gauge invariant action and the compensators to have dynamics we add the following terms in the action:

- Counter terms (they cancel the change of the initial action)

$$S_c = \int d^8 z \left\{ -a_2 \left(D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) B_{\alpha(s-1)\dot{\alpha}(s-1)} \right. \\ \left. + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \right\} , \quad (11)$$

- Kinetic energy terms (the most general free action for each of the compensators)

$$S_{k.e} = \int d^8 z \left\{ e B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)} \right. \\ + h_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ + h_2 V^{\alpha(s-1)\dot{\alpha}(s-1)} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ + h_3 V^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\dot{\gamma}\gamma} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\ \left. + h_4 V^{\alpha(s-1)\dot{\alpha}(s-1)} [D_{\alpha_{s-1}}, \bar{D}_{\dot{\alpha}_{s-1}}] [D^\gamma, \bar{D}^{\dot{\gamma}}] V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \right\} , \quad (12)$$

- Interaction terms (in principle there might be interactions among compensators)

$$S_{int.} = \int d^8 z b B^{\alpha(s-1)\dot{\alpha}(s-1)} \left(D^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \right) . \quad (13)$$

Thus the full action look as below and contains only a series of constants to be

determined.

$$\begin{aligned}
S = \int d^8 z \Bigg\{ & -\frac{1}{2} a_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -\frac{1}{2} a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - a_2 \left(D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) B_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + e B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + h_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + h_2 V^{\alpha(s-1)\dot{\alpha}(s-1)} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + h_3 V^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\dot{\gamma}\gamma} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\
& + h_4 V^{\alpha(s-1)\dot{\alpha}(s-1)} \left[D_{\alpha_{s-1}}, \bar{D}_{\dot{\alpha}_{s-1}} \right] \left[D^\gamma, \bar{D}^{\dot{\gamma}} \right] V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\
& + b B^{\alpha(s-1)\dot{\alpha}(s-1)} \left(D^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \right) \Bigg\} .
\end{aligned} \tag{14}$$

The requirement that this action is invariant under the above transformations, will give rise to two Bianchi identities required for the gauge invariance of the action.

$$\begin{aligned}
D^2 \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} + \left[\frac{1}{s!} \right] D_{(\alpha_s} \mathcal{P}_{\alpha(s-1))\dot{\alpha}(s-1)} &= 0 \quad , \\
\bar{D}^{\dot{\alpha}_{s-1}} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} - \left[\frac{1}{s!} \right] \bar{D}^{\dot{\alpha}_{s-1}} D_{(\alpha_s} \mathcal{G}_{\alpha(s-1))\dot{\alpha}(s-1)} &= 0 \quad ,
\end{aligned} \tag{15}$$

where $\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)}$, $\mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1)}$, $\mathcal{G}_{\alpha(s-1)\dot{\alpha}(s-1)}$ are the variations of the action with respect to the superfields $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$, $V_{\alpha(s-1)\dot{\alpha}(s-1)}$, $B_{\alpha(s-1)\dot{\alpha}(s-1)}$. The solution of the first one gives:

$$h_1 = \frac{1}{2} a_1 \quad , \quad h_2 = 0 \quad , \quad h_3 = 0 \quad , \quad h_4 = 0 \quad , \quad b = 0 \quad , \tag{16}$$

and the second one gives:

$$e = -\frac{1}{2} a_2 \quad , \quad b = 0 \quad . \tag{17}$$

So the gauge invariant action is:

$$\begin{aligned}
S = \int d^8 z \Bigg\{ & -\frac{1}{2} a_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -\frac{1}{2} a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - a_2 \left(D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) B_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& - \frac{1}{2} a_2 B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \frac{1}{2} a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \Bigg\} .
\end{aligned} \tag{18}$$

Now we can integrate out the auxiliary superfield B . Using the on-shell equation of motion of $B_{\alpha(s-1)\dot{\alpha}(s-1)}$ and substitute it back in to the action we get:

$$\begin{aligned}
S = \int d^8 z \Bigg\{ & -\frac{1}{2} a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \frac{1}{2} a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \Bigg\} ,
\end{aligned} \tag{19}$$

and this action is invariant under the transformations

$$\begin{aligned}
\delta \Psi_{\alpha(s)\dot{\alpha}(s-1)} &= -D^2 U_{\alpha(s)\dot{\alpha}(s-1)} + \left[\frac{1}{(s-1)!} \right] \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2))} \\
\delta V_{\alpha(s-1)\dot{\alpha}(s-1)} &= D^{\alpha_s} U_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{U}_{\alpha(s-1)\dot{\alpha}(s)} .
\end{aligned} \tag{20}$$

This theory⁹ is equivalent to the theory of S. Kuzenko and A. Sibiryakov [1], once one solves the constraints that appear in their description (as done in [3]). This theory is well studied and it is known to describe on-shell a massless supermultiplet of superspin $Y=s$.

⁹We could have reached the same result if from the very begging we have choosen $c_1 = a_2 = 0$ instead of introducing the auxiliary compensator B .

3.2 The FVdWH-series

Again by observing equation (4) we find that there is another way to arrange things. By setting $K_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{D}^{\dot{\alpha}s} U_{\alpha(s-1)\dot{\alpha}(s)}$ then we find:

$$\begin{aligned}
\delta S = \int d^8 z \big(& -2c_1 D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + a_2 \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \big) D^\beta \bar{D}_{\dot{\alpha}_{s-1}} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \\
& + 2c_2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 D_{\alpha_s} \bar{D}^{\dot{\alpha}s} U_{\alpha(s-1)\dot{\alpha}(s)} - a_1 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} D^2 \bar{D}_{\dot{\alpha}_s} \bar{D}^{\dot{\alpha}s} U_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \left(a_2 \left[\frac{s+1}{s} \right] - a_1 \right) \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D^2 \bar{D}^{\dot{\alpha}s} U_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_1 \left[\frac{s-1}{s} \right] \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} D_{\alpha_{s-1}} \bar{D}_{\dot{\alpha}_s} D^\beta \bar{D}^{\dot{\alpha}s} U_{\beta\alpha(s-2)\dot{\alpha}(s)} \\
& + c.c.
\end{aligned} \tag{21}$$

and this suggest setting

$$a_2 \left[\frac{s+1}{s} \right] = a_1 \quad , \tag{22}$$

so that we find

$$\begin{aligned}
\delta S = \int d^8 z \big(& -2c_1 D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + a_2 \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \big) D^\beta \bar{D}_{\dot{\alpha}_{s-1}} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \\
& + \left(2c_2 D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} - a_1 \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) \bar{D}^{\dot{\beta}s} U_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} \\
& - a_1 \left[\frac{s-1}{s} \right] D^{\alpha_{s-1}} \bar{D}_{\dot{\beta}} D_\beta \bar{\Psi}^{\beta\alpha(s-2)\dot{\beta}\dot{\alpha}(s-1)} \left(\bar{D}^{\dot{\alpha}s} U_{\alpha(s-1)\dot{\alpha}(s)} + c.c. \right) \\
& + c.c.
\end{aligned} \tag{23}$$

In order to minimise the degrees of freedom that we have to introduce and construct a minimal theory we set:

$$-2c_1 = a_2 \quad , \quad 2c_2 = -a_1 \quad , \tag{24}$$

so the change of the action takes the form

$$\begin{aligned}
\delta S = \int d^8 z \big(& a_2 D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} \left[\frac{1}{(s-1)!} \right] \left[D^\beta \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \right) + c.c. \big] \\
& + a_2 \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \left[\frac{1}{(s-1)!} \right] \left[D^\beta \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\beta\alpha(s-1)\dot{\alpha}(s-2)} \right) + c.c. \big] \\
& - a_1 D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \left[\bar{D}^{\dot{\beta}s} U_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} + c.c. \right] \\
& - a_1 \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \left[\bar{D}^{\dot{\beta}s} U_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-1)} + c.c. \right] \\
& - a_1 \left[\frac{s-1}{s} \right] \left(D^{\alpha_{s-1}} \bar{D}_{\dot{\beta}} D_\beta \bar{\Psi}^{\beta\alpha(s-2)\dot{\beta}\dot{\alpha}(s-1)} + c.c. \right) \left[\bar{D}^{\dot{\alpha}s} U_{\alpha(s-1)\dot{\alpha}(s)} + c.c. \right] \quad .
\end{aligned} \tag{25}$$

We introduce two real compensators, $B_{\alpha(s-1)\dot{\alpha}(s-1)}$, $V_{\alpha(s-1)\dot{\alpha}(s-1)}$ with $[B] = 1$, $[V] = 0$ and gauge transformations

$$\delta B_{\alpha(s-1)\dot{\alpha}(s-1)} = \left[\frac{1}{(s-1)!} \right] \left[D^{\alpha_s} \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2))} + \bar{D}^{\dot{\alpha}_s} D_{(\alpha_{s-1}} \bar{\Lambda}_{\alpha(s-2))\dot{\alpha}(s)} \right] \quad , \quad (26)$$

$$\delta V_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{D}^{\dot{\alpha}_s} U_{\alpha(s-1)\dot{\alpha}(s)} + D^{\alpha_s} \bar{U}_{\alpha(s)\dot{\alpha}(s-1)} \quad ,$$

and the gauge transformation of the Ψ superfield is

$$\delta \Psi_{\alpha(s)\dot{\alpha}(s-1)} = \left[\frac{1}{s!} \right] D_{(\alpha_s} \bar{D}^{\dot{\alpha}_s} U_{\alpha(s-1))\dot{\alpha}(s)} + \left[\frac{1}{(s-1)!} \right] \bar{D}_{(\dot{\alpha}_{s-1}} \Lambda_{\alpha(s)\dot{\alpha}(s-2))} \quad . \quad (27)$$

Hence we have to add a few terms to the action

- Counter terms (they cancel the change of the initial action)

$$\begin{aligned} S_c = \int d^8 z \Big\{ & - \left[\frac{s}{s+1} \right] a_1 \left(D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) B_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + a_1 \left[\frac{s-1}{s} \right] \left(\bar{D}^{\dot{\alpha}_{s-1}} D_{\beta} \bar{D}_{\dot{\beta}} \Psi^{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} + c.c. \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \quad , \end{aligned} \quad (28)$$

- Kinetic energy (the most general action for each of the compensators)

$$\begin{aligned} S_{k.e} = \int d^8 z \quad & e B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + h_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + h_2 V^{\alpha(s-1)\dot{\alpha}(s-1)} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + h_3 V^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\gamma\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\ & + h_4 V^{\alpha(s-1)\dot{\alpha}(s-1)} \left[D_{\alpha_{s-1}}, \bar{D}_{\dot{\alpha}_{s-1}} \right] \left[D^\gamma, \bar{D}^{\dot{\gamma}} \right] V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \quad , \end{aligned} \quad (29)$$

- Interaction terms (in principle there might be interactions among compensators)

$$S_{int} = \int d^8 z b B^{\alpha(s-1)\dot{\alpha}(s-1)} \left(D^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \right) \quad . \quad (30)$$

Therefore the full action is

$$\begin{aligned}
S = \int d^8 z \Big\{ & -\frac{1}{2} \left[\frac{s}{s+1} \right] a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& -\frac{1}{2} a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \left[\frac{s}{s+1} \right] a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - \left[\frac{s}{s+1} \right] a_1 \left(D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) B_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \left[\frac{s-1}{s} \right] a_1 \left(\bar{D}^{\dot{\alpha}_{s-1}} D_{\beta} \bar{D}_{\dot{\beta}} \Psi^{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} + c.c. \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + e B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + h_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + h_2 V^{\alpha(s-1)\dot{\alpha}(s-1)} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + h_3 V^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\gamma\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\
& + h_4 V^{\alpha(s-1)\dot{\alpha}(s-1)} \left[D_{\alpha_{s-1}}, \bar{D}_{\dot{\alpha}_{s-1}} \right] \left[D^\gamma, \bar{D}^{\dot{\gamma}} \right] V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\
& + b B^{\alpha(s-1)\dot{\alpha}(s-1)} \left(D^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \right) \Big\} .
\end{aligned} \tag{31}$$

The invariance of this action under the gauge transformations is guaranteed by the following two Bianchi Identities (derived as before):

$$\begin{aligned}
\left[\frac{1}{s!} \right] \bar{D}_{(\dot{\alpha}_s} D^{\alpha_s} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1))} - \left[\frac{1}{s!} \right] \bar{D}_{(\dot{\alpha}_s} \mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1))} &= 0 , \\
\bar{D}^{\dot{\alpha}_{s-1}} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} - \left[\frac{1}{s!} \right] \bar{D}^{\dot{\alpha}_{s-1}} D_{(\alpha_s} \mathcal{G}_{\alpha(s-1)\dot{\alpha}(s-1))} &= 0 ,
\end{aligned} \tag{32}$$

where $\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)}$, $\mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1)}$, $\mathcal{G}_{\alpha(s-1)\dot{\alpha}(s-1)}$ are the variations of the action with respect to the superfields $\Psi_{\alpha(s)\dot{\alpha}(s-1)}$, $V_{\alpha(s-1)\dot{\alpha}(s-1)}$, $B_{\alpha(s-1)\dot{\alpha}(s-1)}$.

The solution of the first one gives:

$$\begin{aligned}
h_1 &= \left[\frac{1}{2s} \right] a_1 , & h_2 &= - \left[\frac{s-1}{2s} \right] a_1 , \\
h_3 &= \left[\frac{(2s-1)(s-1)}{(2s)^2} \right] a_1 , & h_4 &= \left[\frac{s-1}{(2s)^2} \right] a_1 , \\
b &= -a_1 ,
\end{aligned} \tag{33}$$

and the second one has as a solution:

$$e = -\frac{1}{2} \left[\frac{s}{s+1} \right] a_1 \quad , \quad b = -a_1 \quad . \quad (34)$$

Thus the action takes the form

$$\begin{aligned} S = \int d^8 z \Big\{ & -\frac{1}{2} \left[\frac{s}{s+1} \right] a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ & -\frac{1}{2} a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\ & + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\ & + \left[\frac{s}{s+1} \right] a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\ & - \left[\frac{s}{s+1} \right] a_1 \left(D_{\alpha_s} \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) B_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + a_1 \left(D_{\alpha_s} \bar{D}^2 \Psi^{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}_{\dot{\alpha}_s} D^2 \bar{\Psi}^{\alpha(s-1)\dot{\alpha}(s)} \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + \left[\frac{s-1}{s} \right] a_1 \left(\bar{D}^{\dot{\alpha}_{s-1}} D_{\beta} \bar{D}_{\dot{\beta}} \Psi^{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} + c.c. \right) V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & - \frac{1}{2} \left[\frac{s}{s+1} \right] a_1 B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + \frac{1}{2s} a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & - \left[\frac{s-1}{2s} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\ & + \left[\frac{(2s-1)(s-1)}{(2s)^2} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\gamma\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\ & + \left[\frac{s-1}{(2s)^2} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} [D_{\alpha_{s-1}}, \bar{D}_{\dot{\alpha}_{s-1}}] [D^\gamma, \bar{D}^{\dot{\gamma}}] V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\ & \left. - a_1 B^{\alpha(s-1)\dot{\alpha}(s-1)} \left(D^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \right) \right\} . \quad (35) \end{aligned}$$

At this point we can integrate out the auxiliary superfield B by using it's equation of motion

$$\begin{aligned} B_{\alpha(s-1)\dot{\alpha}(s-1)} = & [D^{\alpha_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}] \\ & - \left[\frac{s+1}{s} \right] \left(D^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \right) , \quad (36) \end{aligned}$$

and substituting this result back in (35) yields

$$\begin{aligned}
S = \int d^8 z \Big\{ & -\frac{1}{2} a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \Psi^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_1 \{ \bar{D}^2, D_{\alpha_s} \} \Psi^{\alpha(s)\dot{\alpha}(s-1)} V_{\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& + \left[\frac{s-1}{s} \right] a_1 \bar{D}^{\dot{\alpha}_{s-1}} D_{\beta} \bar{D}_{\dot{\beta}} \Psi^{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)} V_{\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& + \left[\frac{s+2}{2s} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \left[\frac{1}{s} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + \left[\frac{(2s-1)(s-1)}{(2s)^2} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} \partial^{\gamma\dot{\gamma}} V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \\
& + \left[\frac{s-1}{(2s)^2} \right] a_1 V^{\alpha(s-1)\dot{\alpha}(s-1)} [D_{\alpha_{s-1}}, \bar{D}_{\dot{\alpha}_{s-1}}] [D^\gamma, \bar{D}^{\dot{\gamma}}] V_{\gamma\alpha(s-2)\dot{\gamma}\dot{\alpha}(s-2)} \Big\} .
\end{aligned} \tag{37}$$

Calculating variations with respect to Ψ and V in this action we can define the following superfields

$$\begin{aligned}
\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} = & -a_1 \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} + \frac{a_1}{s!} \bar{D}^{\dot{\alpha}_s} D_{(\alpha_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + \frac{a_1}{s!} \bar{D}^2 D_{(\alpha_s} V_{\alpha(s-1)\dot{\alpha}(s-1)} + \frac{a_1}{s!} D_{(\alpha_s} \bar{D}^2 V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& - \frac{s-1}{s!s!} a_1 \bar{D}_{(\dot{\alpha}_{s-1}} D_{(\alpha_s} \bar{D}^{\dot{\beta}} V_{\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)}) ,
\end{aligned} \tag{38}$$

$$\begin{aligned}
\mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1)} = & -a_1 D^{\alpha_s} \bar{D}^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} - a_1 \bar{D}^{\dot{\alpha}_s} D^2 \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& - a_1 \bar{D}^2 D^{\alpha_s} \Psi_{\alpha(s)\dot{\alpha}(s-1)} - a_1 D^2 \bar{D}^{\dot{\alpha}_s} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \\
& + a_1 \frac{s-1}{s!} \bar{D}_{(\dot{\alpha}_{s-1}} D^\beta \bar{D}^{\dot{\beta}} \Psi_{\beta\alpha(s-1)\dot{\beta}\dot{\alpha}(s-2)}) \\
& + a_1 \frac{s-1}{s!} D_{(\alpha_{s-1}} \bar{D}^{\dot{\beta}} D^\beta \bar{\Psi}_{\beta\alpha(s-2)\dot{\beta}\dot{\alpha}(s-1)} \\
& + a_1 \frac{s+2}{s} D^\gamma \bar{D}^2 D_\gamma V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + a_1 \frac{2}{s} \square V_{\alpha(s-1)\dot{\alpha}(s-1)} \\
& + a_1 \frac{(2s-1)(s-1)}{2s!^2} \partial_{(\alpha_{s-1}(\dot{\alpha}_{s-1}} \partial^{\beta\dot{\beta}} V_{\beta\alpha(s-2)\dot{\beta}\dot{\alpha}(s-2)}) \\
& + a_1 \frac{s-1}{2s!^2} [D_{(\alpha_{s-1}}, \bar{D}_{(\dot{\alpha}_{s-1}}] [D^\beta, \bar{D}^{\dot{\beta}}] V_{\beta\alpha(s-2)\dot{\beta}\dot{\alpha}(s-2)}) ,
\end{aligned} \tag{39}$$

and they satisfy the Bianchi Identities for this final action

$$\begin{aligned}\bar{D}^{\dot{\alpha}_{s-1}} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} &= 0 \quad , \\ \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} D^{\alpha_s} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1))} - \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} \mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1))} &= 0 \quad .\end{aligned}\tag{40}$$

Furthermore we can prove that they also satisfy the following identity:

$$\begin{aligned}\bar{D}^{\dot{\alpha}_{2s}} \bar{\mathcal{W}}_{\dot{\alpha}(2s)} &= -i \frac{2s}{a_1} \partial^{\alpha_s}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1))} \\ &\quad - \frac{s}{a_1} \bar{D}^2 \partial^{\alpha_{s-1}}_{(\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\mathcal{T}}_{\alpha(s-1)\dot{\alpha}(s)} \\ &\quad + \frac{s}{a_1} \bar{D}_{(\dot{\alpha}_{2s-1}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-2}} \dots \partial^{\alpha_1}_{\dot{\alpha}_s} \mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1))} \quad ,\end{aligned}\tag{41}$$

where the anti-chiral superfield (i.e. $D_\beta \bar{\mathcal{W}}_{\dot{\alpha}(2s)} = 0$) is given by

$$\bar{\mathcal{W}}_{\dot{\alpha}(2s)} = D^2 \bar{D}_{(\dot{\alpha}_{2s}} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{2s-1}} \dots \partial^{\alpha_1}_{\dot{\alpha}_{s+1}} \bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)} \quad .\tag{42}$$

So this theory has an irreducible multiplet propagating on-shell with superspin $Y = s$. Now we can check if these are the only degrees of freedom propagating.

Expanding the superfields Ψ , V into components and using their gauge transformations we find that some components have purely algebraic transformations and therefore can be gauged away. In detail¹⁰:

For Bosons		For Fermions	
Component	Gauged away by	Component	Gauged away by
$V_{\alpha(s-1)\dot{\alpha}(s-1)}^{(0,0)}$	$Re \left[U_{\alpha(s-1)\dot{\alpha}(s-1)}^{(0,1)(A)} \right]$	$V_{\alpha(s-1)\dot{\alpha}(s)}^{(0,1)(S)}$	$U_{\alpha(s-1)\dot{\alpha}(s)}^{(0,2)}$
$V_{\alpha(s)\dot{\alpha}(s)}^{(1,1)(S,S)}$	$Re \left[U_{\alpha(s)\dot{\alpha}(s)}^{(1,2)(S)} \right]$	$V_{\alpha(s-2)\dot{\alpha}(s-1)}^{(1,0)(A)}$	$U_{\alpha(s-2)\dot{\alpha}(s-1)}^{(1,1)(A,A)}$
$V_{\alpha(s-2)\dot{\alpha}(s)}^{(1,1)(A,S)}$	$U_{\alpha(s-2)\dot{\alpha}(s)}^{(1,2)(A)}$	$V_{\alpha(s-1)\dot{\alpha}(s)}^{(2,1)(S)}$	$U_{\alpha(s-1)\dot{\alpha}(s)}^{(2,2)}$
$V_{\alpha(s-1)\dot{\alpha}(s-1)}^{(2,0)}$	$U_{\alpha(s-1)\dot{\alpha}(s-1)}^{(2,1)(A)}$	$\Psi_{\alpha(s)\dot{\alpha}(s-1)}^{(0,0)}$	$\Lambda_{\alpha(s)\dot{\alpha}(s-1)}^{(0,1)(S)}$
$\Psi_{\alpha(s+1)\dot{\alpha}(s-1)}^{(1,0)(S)}$	$\Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}^{(1,1)(S,S)}$	$\Psi_{\alpha(s)\dot{\alpha}(s-1)}^{(2,0)}$	$\Lambda_{\alpha(s)\dot{\alpha}(s-1)}^{(2,1)(S)}$
$\Psi_{\alpha(s-1)\dot{\alpha}(s-1)}^{(1,0)(A)}$	$\Lambda_{\alpha(s-1)\dot{\alpha}(s-1)}^{(1,1)(A,S)}$	$\Psi_{\alpha(s+1)\dot{\alpha}(s-2)}^{(1,1)(S,A)}$	$\Lambda_{\alpha(s+1)\dot{\alpha}(s-2)}^{(1,2)(S)}$
$Im \left[\Psi_{\alpha(s)\dot{\alpha}(s)}^{(0,1)(S)} \right]$	$Im \left[U_{\alpha(s)\dot{\alpha}(s)}^{(1,2)(S)} \right]$	$\Psi_{\alpha(s-1)\dot{\alpha}(s-2)}^{(1,1)(A,A)}$	$\Lambda_{\alpha(s-1)\dot{\alpha}(s-2)}^{(1,2)(A)}$
$\Psi_{\alpha(s)\dot{\alpha}(s-2)}^{(2,1)(A)}$	$\Lambda_{\alpha(s)\dot{\alpha}(s-2)}^{(2,2)}$		

¹⁰The definition of symmetric and antisymmetric pieces of a field is the following

$$\Phi_{\gamma\alpha(s-1)} = \Phi_{\gamma\alpha(s-1)}^{(S)} + \frac{s-1}{s!} C_{\gamma(\alpha_{s-1}} \Phi_{\alpha(s-2))}^{(A)} \quad , \quad \Phi_{\gamma\alpha(s-1)}^{(S)} = \frac{1}{s!} \Phi_{(\gamma\alpha(s-1))}^{(S)} \quad , \quad \Phi_{\alpha(s-2)}^{(A)} = C^{\gamma\alpha_{s-1}} \Phi_{\gamma\alpha(s-1)}$$

Furthermore the notation $\Phi^{(m,n)}$ represents the $\theta^m \bar{\theta}^n$ component in the taylor series of the superfield Φ

So in the Wess-Zumino gauge the two superfields take the forms

$$\begin{aligned}
V_{\alpha(s-1)\dot{\alpha}(s-1)} &= \left[\frac{2(s-1)}{(s-1)!^2} \right] \theta_{(\alpha_{s-1}} \bar{\theta}_{(\dot{\alpha}_{s-1}} h_{\alpha(s-2))\dot{\alpha}(s-2))} \\
&+ \left[\frac{\sqrt{2}}{(s-1)!} \right] \theta^2 \bar{\theta}_{(\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2))} \\
&- \left[\frac{\sqrt{2}}{(s-1)!} \right] \theta_{(\alpha_{s-1}} \bar{\theta}^2 \bar{\psi}_{\alpha(s-2))\dot{\alpha}(s-1)} + \theta^2 \bar{\theta}^2 P_{\alpha(s-1)\dot{\alpha}(s-1)} \quad ,
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\Psi_{\alpha(s)\dot{\alpha}(s-1)} &= \bar{\theta}^{\dot{\alpha}s} h_{\alpha(s)\dot{\alpha}(s)} + \sqrt{2} \bar{\theta}^2 \psi_{\alpha(s)\dot{\alpha}(s-1)} + \sqrt{2} \theta^{\alpha s+1} \bar{\theta}^{\dot{\alpha}s} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
&+ \left[\frac{1}{s!} \right] \theta_{(\alpha_s} \bar{\theta}^{\dot{\alpha}s} \left[\lambda_{\alpha(s-1))\dot{\alpha}(s)} - \frac{\sqrt{2}s}{s+1} \bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)} \right] \\
&+ \theta^2 \bar{\theta}^{\dot{\alpha}s} Y_{\alpha(s)\dot{\alpha}(s)} \\
&+ \bar{\theta}^2 \theta^{\alpha s+1} \left[t_{\alpha(s+1)\dot{\alpha}(s-1)} + i \frac{3}{2(s+1)!} \partial_{(\alpha_{s+1}} \dot{\alpha}_s h_{\alpha(s))\dot{\alpha}(s)} \right] \\
&+ \left[\frac{s}{(s+1)!} \right] \bar{\theta}^2 \theta_{(\alpha_s} \left[M_{\alpha(s-1))\dot{\alpha}(s-1)} + \frac{(s+1)^2}{s(2s+1)} P_{\alpha(s-1))\dot{\alpha}(s-1)} \right] \\
&+ i \left[\frac{s}{(s+1)!} \right] \bar{\theta}^2 \theta_{(\alpha_s} \left[N_{\alpha(s-1))\dot{\alpha}(s-1)} + \frac{2s-1}{2} \partial^{\gamma\dot{\gamma}} h_{\gamma\alpha(s-1))\dot{\gamma}\dot{\alpha}(s-1)} \right. \\
&\quad \left. - \frac{(s+1)^2(s-1)}{s!} \partial_{\alpha_{s-1}(\dot{\alpha}_{s-1}} h_{\alpha(s-2))\dot{\alpha}(s-2))} \right] \\
&+ \theta^2 \bar{\theta}^2 \left[\chi_{\alpha(s)\dot{\alpha}(s-1)} + \frac{i}{\sqrt{2}} \frac{s-1}{s+1} \partial^{\alpha s+1 \dot{\alpha}s} \psi_{\alpha(s+1)\dot{\alpha}(s)} \right. \\
&\quad \left. - \frac{i}{\sqrt{2}} \frac{s(s-1)}{(s+1)^2 s!} \partial_{(\alpha_s} \dot{\alpha}_s \bar{\psi}_{\alpha(s-1))\dot{\alpha}(s)} - \frac{i}{2(s+1)!} \partial_{(\alpha_s} \dot{\alpha}_s \lambda_{\alpha(s-1))\dot{\alpha}(s)} \right] \quad ,
\end{aligned} \tag{44}$$

where the components t , M , N , P , Y , λ , χ will be shown to be auxiliary fields. All the others are symmetric in all undotted and dotted indices separately and the components $h_{\alpha(s)\dot{\alpha}(s)}$, $h_{\alpha(s-2)\dot{\alpha}(s-2)}$ are real. The component action for all the bosons is:

$$\begin{aligned}
S_{Bosons} &= \int d^4x \left\{ -2a_1 h^{\alpha(s)\dot{\alpha}(s)} \square h_{\alpha(s)\dot{\alpha}(s)} \right. \\
&\quad + s a_1 h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial^{\dot{\gamma}\gamma} h_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \\
&\quad - [2s(s-1)] a_1 h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial_{\alpha_{s-1} \dot{\alpha}_{s-1}} h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad \left. + [2s(2s-1)] a_1 h^{\alpha(s-2)\dot{\alpha}(s-2)} \square h_{\alpha(s-2)\dot{\alpha}(s-2)} \right\}
\end{aligned}$$

$$\begin{aligned}
& + [s(s-2)^2] a_1 h^{\alpha(s-2)\dot{\alpha}(s-2)} \partial_{\alpha_{s-2}\dot{\alpha}_{s-2}} \partial^{\dot{\gamma}\gamma} h_{\gamma\alpha(s-3)\dot{\gamma}\dot{\alpha}(s-3)} \\
& + \frac{1}{2} a_1 t^{\alpha(s+1)\dot{\alpha}(s-1)} t_{\alpha(s+1)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \left[\frac{2s+1}{s+1} \right] M^{\alpha(s-1)\dot{\alpha}(s-1)} M_{\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& - a_1 \left[\frac{(s+1)^3}{s^2(2s+1)} \right] P^{\alpha(s-1)\dot{\alpha}(s-1)} P_{\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& + a_1 \left[\frac{1}{s+1} \right] N^{\alpha(s-1)\dot{\alpha}(s-1)} N_{\alpha(s-1)\dot{\alpha}(s-1)} + c.c. \\
& - a_1 Y^{\alpha(s)\dot{\alpha}(s)} \bar{Y}_{\alpha(s)\dot{\alpha}(s)} ,
\end{aligned} \tag{45}$$

and using the equations of motion for the auxiliary fields

$$\begin{aligned}
M_{\alpha(s-1)\dot{\alpha}(s-1)} &= 0 , \quad N_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 , \quad P_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 , \\
t_{\alpha(s+1)\dot{\alpha}(s-1)} &= 0 , \quad Y_{\alpha(s)\dot{\alpha}(s)} = 0 ,
\end{aligned} \tag{46}$$

we obtain

$$\begin{aligned}
S_{Bosons} &= \int d^4x \left\{ -2a_1 h^{\alpha(s)\dot{\alpha}(s)} \square h_{\alpha(s)\dot{\alpha}(s)} + s a_1 h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial^{\dot{\gamma}\gamma} h_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right. \\
&\quad - [2s(s-1)] a_1 h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad + [2s(2s-1)] a_1 h^{\alpha(s-2)\dot{\alpha}(s-2)} \square h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad \left. + [s(s-2)^2] a_1 h^{\alpha(s-2)\dot{\alpha}(s-2)} \partial_{\alpha_{s-2}\dot{\alpha}_{s-2}} \partial^{\dot{\gamma}\gamma} h_{\gamma\alpha(s-3)\dot{\gamma}\dot{\alpha}(s-3)} \right\} ,
\end{aligned} \tag{47}$$

and upon by setting $a_1 = -\frac{1}{2}$ we find

$$\begin{aligned}
S_{Bosons} &= \int d^4x \left\{ h^{\alpha(s)\dot{\alpha}(s)} \square h_{\alpha(s)\dot{\alpha}(s)} - \frac{s}{2} h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial^{\dot{\gamma}\gamma} h_{\gamma\alpha(s-1)\dot{\gamma}\dot{\alpha}(s-1)} \right. \\
&\quad + [s(s-1)] h^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha_s \dot{\alpha}_s} \partial_{\alpha_{s-1}\dot{\alpha}_{s-1}} h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad - [s(2s-1)] h^{\alpha(s-2)\dot{\alpha}(s-2)} \square h_{\alpha(s-2)\dot{\alpha}(s-2)} \\
&\quad \left. - \left[\frac{s(s-2)^2}{2} \right] h^{\alpha(s-2)\dot{\alpha}(s-2)} \partial_{\alpha_{s-2}\dot{\alpha}_{s-2}} \partial^{\dot{\gamma}\gamma} h_{\gamma\alpha(s-3)\dot{\gamma}\dot{\alpha}(s-3)} \right\} ,
\end{aligned} \tag{48}$$

which is the Fronsdal action for a propagating massless spin- s bosonic field.

The fermionic piece of the action is:

$$\begin{aligned}
S_{Fermions} &= \int d^4x \left\{ -i2a_1 \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}}_{\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \right. \\
&\quad + i2 \left[\frac{2s+1}{(s+1)^2} \right] a_1 \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s}_{\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
&\quad \left. + i2a_1 \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \right\}
\end{aligned}$$

$$\begin{aligned}
& -i2 \left[\frac{s}{s+1} \right] a_1 \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& + i2a_1 \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c. \\
& + \left[\frac{s+1}{s} \right] \bar{\lambda}^{\alpha(s)\dot{\alpha}(s-1)} \chi_{\alpha(s)\dot{\alpha}(s-1)} + c.c.
\end{aligned} \tag{49}$$

so that using the equations of motions for the auxiliary fields χ , λ

$$\lambda_{\alpha(s-1)\dot{\alpha}(s)} = 0 \quad , \quad \chi_{\alpha(s)\dot{\alpha}(s-1)} = 0 \quad , \tag{50}$$

and setting the value of $a_1 = -\frac{1}{2}$, we get the final fermionic action

$$\begin{aligned}
S_{Fermions} = \int d^4x \Big\{ & i \bar{\psi}^{\alpha(s)\dot{\alpha}(s+1)} \partial^{\alpha_{s+1}}_{\dot{\alpha}_{s+1}} \psi_{\alpha(s+1)\dot{\alpha}(s)} \\
& - i \left[\frac{2s+1}{(s+1)^2} \right] \bar{\psi}^{\alpha(s-1)\dot{\alpha}(s)} \partial^{\alpha_s}_{\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} \\
& - i \bar{\psi}^{\alpha(s-2)\dot{\alpha}(s-1)} \partial^{\alpha_{s-1}}_{\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} \\
& + i \left[\frac{s}{s+1} \right] \psi^{\alpha(s+1)\dot{\alpha}(s)} \partial_{\alpha_{s+1}\dot{\alpha}_s} \psi_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
& - i \psi^{\alpha(s)\dot{\alpha}(s-1)} \partial_{\alpha_s\dot{\alpha}_{s-1}} \psi_{\alpha(s-1)\dot{\alpha}(s-2)} + c.c.
\end{aligned} \tag{51}$$

This is the Fronsdal action for a propagating massless spin- $(s+1/2)$. Therefore we conclude that only an irreducible supermultiplet propagates on-shell and therefore the action (37) describes a massless integer superspin $Y = s$.

The counting of the off-shell bosonic and fermionic degrees of freedom for the action including all the auxiliary fields is:

Component Field(s)	Bosonic	Fermionic
$h_{\alpha(s)\dot{\alpha}(s)} / h_{\alpha(s-2)\dot{\alpha}(s-2)}$	$s^2 + 2$	
$\psi_{\alpha(s+1)\dot{\alpha}(s)} / \psi_{\alpha(s)\dot{\alpha}(s-1)} / \psi_{\alpha(s-1)\dot{\alpha}(s-2)}$		$4(s^2 + s + 1)$
$P_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	
$\lambda_{\alpha(s-1)\dot{\alpha}(s)}$		$2s(s+1)$
$Y_{\alpha(s)\dot{\alpha}(s)}$	$2(s+1)^2$	
$t_{\alpha(s+1)\dot{\alpha}(s-1)}$	$2(s+2)s$	
$M_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	
$N_{\alpha(s-1)\dot{\alpha}(s-1)}$	s^2	
$\chi_{\alpha(s)\dot{\alpha}(s-1)}$		$2s(s+1)$
	$8s^2 + 8s + 4$	$8s^2 + 8s + 4$

For each case we have verified the existence of field strength superfields $\mathcal{W}_{\alpha(2s)}$, $\mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1)}$ and $\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)}$ which occur for both the KS-series and the FVdWH-series.

4 Considering the $s = 1$ case

For the special case of $s = 1$, the most general action takes the form:

$$S = \int d^8z \left\{ a_1 \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + a_2 \Psi^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} \right. \\ \left. + [c_1 \Psi^\alpha D^2 \Psi_\alpha + c_2 \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c.] \right\} . \quad (52)$$

Also in this case, the gauge parameter $\Lambda_{\alpha(s)\dot{\alpha}(s-2)}$, in order to survive, must be modified to $\bar{D}^{\dot{\alpha}s-1} \Lambda_{\alpha(s)\dot{\alpha}(s-1)}$. So the most general gauge transformation allowed in the $s = 1$ is:

$$\delta \Psi_\alpha = D_\alpha K + \bar{D}^2 \Lambda_\alpha . \quad (53)$$

The change of the general action (52) under this transformation is:

$$\delta S = \int d^8z \left(-2c_1 D_\alpha \Psi^\alpha + a_2 \bar{D}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}} \right) D^\beta \bar{D}^2 \Lambda_\beta + c.c. \\ + 2c_2 \Psi^\alpha \bar{D}^2 D_\alpha K + c.c. \\ - a_1 \bar{\Psi}^{\dot{\alpha}} D^2 \bar{D}_{\dot{\alpha}} K + c.c. \\ + (2a_2 - a_1) \bar{\Psi}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^2 K + c.c. \quad (54)$$

At this point, we can do a very useful observation. All the propagating degrees of freedom required for the formulation of a massless $Y = 1$ theory, can be included in the main superfield Ψ_α . To verify that, just look the Taylor expansion of the superfields (44) and (43). So as a consequence, either we have to make the change of the action to vanish or add purely auxiliary compensators.

4.1 A) $K = D^\alpha U_\alpha$

For this choice of K we find the following action

$$S = \int d^8z \left\{ -\frac{1}{2} a_1 \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \right. \\ + a_1 \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} \\ + a_1 (D_\alpha \bar{D}^2 \Psi^\alpha + \bar{D}_{\dot{\alpha}} D^2 \bar{\Psi}^{\dot{\alpha}}) V \\ \left. + \frac{1}{2} a_1 V D^\gamma \bar{D}^2 D_\gamma V \right\} , \quad (55)$$

and this action is invariant under the transformations

$$\begin{aligned}\delta\Psi_\alpha &= -D^2U_\alpha + \bar{D}^2\Lambda_\alpha \quad , \\ \delta V &= D^\alpha U_\alpha + \bar{D}^{\dot{\alpha}}\bar{U}_{\dot{\alpha}} \quad ,\end{aligned}\tag{56}$$

and the Bianchi Identities are:

$$\begin{aligned}D^2\mathcal{T}_\alpha + D_\alpha\mathcal{P} &= 0 \quad , \\ \bar{D}^2\mathcal{T}_\alpha &= 0 \quad .\end{aligned}\tag{57}$$

where

$$\begin{aligned}\mathcal{T}_\alpha &= -a_1\bar{D}^2\Psi_\alpha + a_1\bar{D}^{\dot{\alpha}}D_\alpha\bar{\Psi}_{\dot{\alpha}} + a_1\bar{D}^2D_\alpha V \quad , \\ \mathcal{P} &= a_1D^\gamma\bar{D}^2D_\gamma V - a_1\left(D^\alpha\bar{D}^2\Psi_\alpha + \bar{D}^{\dot{\alpha}}D^2\bar{\Psi}_{\dot{\alpha}}\right) \quad .\end{aligned}\tag{58}$$

This is the $s = 1$ limit of (19).

On-shell the propagating degrees of freedom of superfield V are gauged away completely and the only thing that survives is the $Y = 1$ supermultiplet. This can be visualized by the following argument. There is a gauge where $V = 0$. This happens for $U_\alpha = i\bar{D}^2D_\alpha L$. Working in this gauge the V superfield vanishes from the action which becomes:

$$S = \int d^8z \left\{ -\frac{1}{2}a_1\Psi^\alpha\bar{D}^2\Psi_\alpha - \frac{1}{2}a_1\bar{\Psi}^{\dot{\alpha}}D^2\bar{\Psi}_{\dot{\alpha}} + a_1\Psi^\alpha\bar{D}^{\dot{\alpha}}D_\alpha\bar{\Psi}_{\dot{\alpha}} \right\} \quad ,\tag{59}$$

and is invariant under the transformation

$$\delta\Psi_\alpha = D^2\bar{D}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}L + \bar{D}^2\Lambda_\alpha \quad .\tag{60}$$

This is the formulation suggested first by Fradkin and Vasiliev in [4] and de Wit and van Holten in [5] at the component level and in [6] for a superfield description. This last work also made the observation that this formulation was distinct from an earlier off-shell description of the $Y = 1$ supermultiplet [7].

4.2 B) $K = \bar{D}^{\dot{\alpha}} U_{\dot{\alpha}}$

For this choice of K we get the action:

$$\begin{aligned}
S = \int d^8 z \Big\{ & -\frac{1}{2} a_1 \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \\
& + a_1 \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} \\
& + a_1 (D_\alpha \bar{D}^2 \Psi^\alpha + \bar{D}_{\dot{\alpha}} D^2 \bar{\Psi}^{\dot{\alpha}}) V \\
& + a_1 (\bar{D}^2 D_\alpha \Psi^\alpha + D^2 \bar{D}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}}) V \\
& + \frac{3}{2} a_1 V D^\gamma \bar{D}^2 D_\gamma V \\
& + a_1 V \square V \Big\} ,
\end{aligned} \tag{61}$$

and it is invariant under the gauge transformations

$$\begin{aligned}
\delta \Psi_\alpha &= D_\alpha \bar{D}^{\dot{\alpha}} U_{\dot{\alpha}} + \bar{D}^2 \Lambda_\alpha , \\
\delta V &= \bar{D}^{\dot{\alpha}} U_{\dot{\alpha}} + D^\alpha \bar{U}_\alpha ,
\end{aligned} \tag{62}$$

with Bianchi Identities

$$\begin{aligned}
\bar{D}_{\dot{\alpha}} D^\alpha \mathcal{T}_\alpha - \bar{D}_{\dot{\alpha}} \mathcal{P} &= 0 \\
\bar{D}^2 \mathcal{T}_\alpha &= 0
\end{aligned} \tag{63}$$

with

$$\begin{aligned}
\mathcal{T}_\alpha &= -a_1 \bar{D}^2 \Psi_\alpha + a_1 \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + a_1 \{ \bar{D}^2, D_\alpha \} V , \\
\mathcal{P} &= 3a_1 D^\gamma \bar{D}^2 D_\gamma V + 2a_1 \square V - a_1 \left[\{ \bar{D}^2, D^\alpha \} \Psi_\alpha + \{ D^2, \bar{D}^{\dot{\alpha}} \} \bar{\Psi}_{\dot{\alpha}} \right] .
\end{aligned} \tag{64}$$

Like before there is no component of V surviving on-shell. The only propagating sub-multiplet is the $Y = 1$ supermultiplet. This is the $s = 1$ limit of (37).

4.3 C) One more thing...

$$K = \bar{K}, \quad \Lambda_\alpha = i D_\alpha U, \quad U = \bar{U}$$

For the special case of $s = 1$ there is one more possibility.

If $K = \bar{K}$, $\Lambda_\alpha = i D_\alpha U$, $U = \bar{U}$ the change of the action becomes

$$\begin{aligned}
\delta S = \int d^8 z \, i \, & (-2c_1 - a_2) D_\alpha \Psi^\alpha D^\beta \bar{D}^2 D_\beta U + c.c. \\
& + (2c_2 - a_1) \Psi^\alpha \bar{D}^2 D_\alpha K + c.c. \\
& + (2a_2 - a_1) \bar{\Psi}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^2 K + c.c.
\end{aligned} \tag{65}$$

which suggests that by choosing:

$$-2c_1 = a_2 \quad , \quad 2c_2 = a_1 \quad , \quad 2a_2 = a_1 \quad . \quad (66)$$

we get:

$$S = \int d^8z \left\{ + a_1 \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + \frac{1}{2} a_1 \Psi^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} \right. \\ \left. + a_1 \left[-\frac{1}{4} \Psi^\alpha D^2 \Psi_\alpha + \frac{1}{2} \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \right] \right\} \quad (67)$$

This is invariant under the transformation

$$\delta \Psi_\alpha = D_\alpha K + i \bar{D}^2 D_\alpha U \quad , \quad (68)$$

with $K = \bar{K}$, $U = \bar{U}$. This formulation was considered by Ogievetsky and Sokatchev in [7] and was noted later in [6] and describes a massless $Y = 1$ supermultiplet. Referring back to this work, it can be seen that the following spectrum of fields was presented.

Component Field(s)	Bosonic	Fermionic
$A_{\alpha\dot{\alpha}}$	3	
$\psi_{\alpha\beta\dot{\alpha}} / \psi_{\alpha\dot{\alpha}}$		12
P	1	
$\lambda_{\dot{\alpha}}$		4
$Y_{\alpha\dot{\alpha}}$	8	
$t_{\alpha\beta}$	6	
M	1	
N	1	
$\chi_{\alpha(s)}$		4
	20	20

A brief comparison between this table and the previous reveals a surprise, but a very satisfying one.

To take the limit of the table at the bottom of page seventeen we begin by substituting $s = 1$. Upon this substitution, any field with a subscript that takes a 0-value means that index does not appear on the field. For any field with a subscript that takes a value < 0 means that field does not appear at all. When these rules are applied and the value $s = 1$ is used in the second and third columns, the two table match perfectly¹¹!

¹¹We anticipated this in naming the fields that appear in the expansions on page fifteen.

In other words, the matter gravitino multiplet described as “the (3/2,1) superfield of O(2) supergravity” is the lowest member of the FVdWH-series tower of higher spin multiplets of such theories. If this is true that means that there must be a duality between the two theories in case B) and case C). The answer is yes, these theories are dual to each other and the duality mechanism is provided by the $s = 1$ limit of (35)

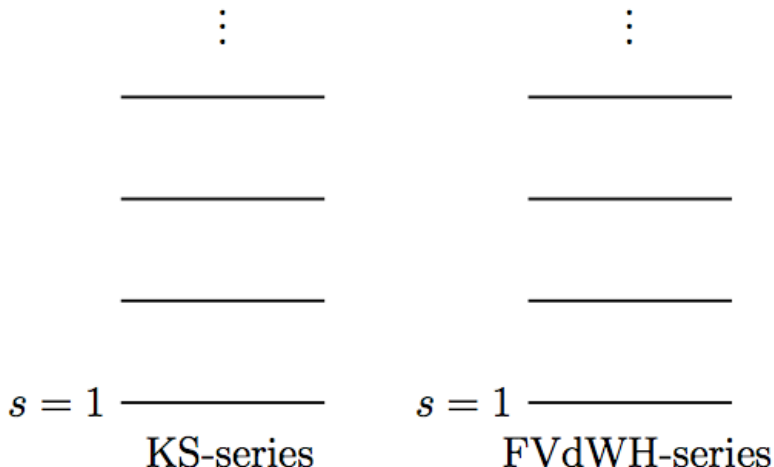
$$\begin{aligned}
S = \int d^8 z \Big\{ & -\frac{1}{4}a_1 \Psi^\alpha D^2 \Psi_\alpha - \frac{1}{2}a_1 \Psi^\alpha \bar{D}^2 \Psi_\alpha + c.c. \\
& + a_1 \Psi^\alpha \bar{D}^{\dot{\alpha}} D_\alpha \bar{\Psi}_{\dot{\alpha}} + \left[\frac{1}{2} \right] a_1 \Psi^\alpha D_\alpha \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} \\
& - \left[\frac{1}{2} \right] a_1 (D_\alpha \Psi^\alpha + \bar{D}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}}) B \\
& + a_1 (D_\alpha \bar{D}^2 \Psi^\alpha + \bar{D}_{\dot{\alpha}} D^2 \bar{\Psi}^{\dot{\alpha}}) V \\
& - \frac{1}{4}a_1 B B + \frac{1}{2}a_1 V D^\gamma \bar{D}^2 D_\gamma V \\
& - a_1 B (D^2 V + \bar{D}^2 V) \Big\} \quad ,
\end{aligned} \tag{69}$$

which is invariant under the transformations

$$\begin{aligned}
\delta \Psi_\alpha &= D_\alpha \bar{D}^{\dot{\alpha}} U_{\dot{\alpha}} - \bar{D}^2 \Lambda_\alpha \quad , \\
\delta V &= \bar{D}^{\dot{\alpha}} U_{\dot{\alpha}} + D^\alpha \bar{U}_\alpha \quad , \\
\delta B &= -D^\alpha \bar{D}^2 \Lambda_\alpha - \bar{D}^{\dot{\alpha}} D^2 \bar{\Lambda}_{\dot{\alpha}} \quad .
\end{aligned} \tag{70}$$

From this point forward there are two choices:

- Choice 1) We can integrate out the auxiliary superfield B as we did in the general case and this will give us (61)
- Choice 2) We can work in a gauge where $B = V = 0$ and this will give (up to a redefinition of the gauge parameter) (67)



In our accompanying work of the half-odd integer case [8], we found that the non-minimal off-shell supergravity theory first discovered by Breitenlohner [9], is the lowest level of an infinite towers of such theories. In this work we have found the same thing for “the (3/2,1) superfield of O(2) supergravity.” This particular off-shell matter gravitino multiplet together with the non-minimal off-shell supergravity multiplet provides a description of 4D, $\mathcal{N} = 2$ supergravity [5, 10]. It is therefore reasonable to expect¹² that this 4D, $\mathcal{N} = 2$ supersymmetry can persist when the B-series ($\mathcal{W}_{\alpha(2s+1)}$, $\mathcal{G}_{\alpha(s)\dot{\alpha}(s)}$ and $\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)}$) and FVdWH-series ($\mathcal{W}_{\alpha(2s)}$, $\mathcal{P}_{\alpha(s-1)\dot{\alpha}(s-1)}$ and $\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)}$) towers are taken together. A similar behavior was observed for alternate towers [11].

“The opposite of a correct statement is a false statement. But the opposite of a profound truth may well be another profound truth.”

– Niels Bohr

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¹²A similar realization of 4D, $\mathcal{N} = 2$ supersymmetry has been found previously [11].

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